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# A note on strong convergence to common fixed points of nonexpansive mappings in Hilbert spaces

Jean-Philippe Chancelier\*

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## Résumé

The aim of this paper is to investigate the links between  $\mathcal{T}_C$ -class algorithms [1], CQ Algorithm [6, 8] and shrinking projection methods [9]. We show that strong convergence of these algorithms are related to coherent  $\mathcal{T}_C$ -class sequences of mapping. Some examples dealing with nonexpansive finite set of mappings and nonexpansive semigroups are given. They extend some existing theorems in [1, 6, 9, 7].

## 1 Introduction

Let  $C$  be a closed convex subset of a Hilbert space  $\mathcal{H}$ . A mapping  $T$  of  $C$  into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by  $\text{Fix}(T)$  the set of fixed points of  $T$ . That is

$$\text{Fix}(T) \stackrel{\text{def}}{=} \{x \in C : Tx = x\}. \quad (1)$$

There are many iterative methods for approximation of common fixed points of a family of nonexpansive mapping in a Hilbert space. In Section 2 we recall the CQ Algorithm [6, 8] (Algorithm 2) associated to a sequence of mappings  $(T_n)_{n \geq 0}$  of  $C$  into itself. The CQ Algorithm when applied to a sequence of mappings of  $\mathcal{H}$  into itself is the same as a Haugazeau method [4] studied in [1, Algorithm 3.1] and applied to  $\mathcal{T}$ -class mappings.

We straightforwardly generalize, in Section 2, the  $\mathcal{T}$ -class to take into account mappings of  $C$  into itself. We denote this new class by the  $\mathcal{T}_C$ -class. Using this extension, the CQ Algorithm (Algorithm 2) coincides with the Haugazeau method (Algorithm 1) and a strong convergence theorem can be obtained by following results from [1]. Note that the convergence theorem is obtained for  $\mathcal{T}_C$ -class sequences which are coherent (Definition 3).

In [9] another algorithm called the shrinking projection method is also studied. One of our aims in this article is to prove that, rephrased in the context

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of  $\mathcal{T}_C$ -class algorithm, the convergence results of this new algorithm (Algorithm 3) is also related to coherent sequences of  $\mathcal{T}_C$ -class mappings. We give in Theorem 6 a strong convergence result of Algorithm 3 for  $\mathcal{T}_C$ -class coherent sequence of mappings. Section 4 is devoted to the proof. The strong convergence of Algorithm 3 is also proved in [9, Theorem 3.3] for sequence of nonexpansive mappings satisfying the NST-condition(I) (Definition 9). It is easy to prove that if  $R$  is a nonexpansive mapping of  $C$  into itself then  $T = (R + Id)/2$  belongs to the  $\mathcal{T}_C$ -class and that a sequence of nonexpansive mappings satisfying the NST-condition(I) is coherent. Thus Theorem 6 extends [9, Theorem 3.3 and Theorem 3.4].

In Section 3 we show that specific sequences of mappings are coherent. Combined with Theorem 6 it can be considered as an extension to some existing theorems in [6, 9, 7].

## 2 The $\mathcal{T}_C$ -class iterative algorithms, CQ algorithm and the shrinking projection method

We first recall here the  $\mathcal{T}$ -class iterative algorithms as defined by H. Bauschke and P. L. Combettes [1].

For  $(x, y) \in \mathcal{H}^2$  and  $S$  a subset of  $\mathcal{H}$ , we define the mapping  $H_S$  as follows :

$$H_S(x, y) \stackrel{\text{def}}{=} \{z \in S \mid \langle z - y, x - y \rangle \leq 0\} . \quad (2)$$

We also define the mapping  $H$  by  $H \stackrel{\text{def}}{=} H_{\mathcal{H}}$ . Note that  $H_S(x, x) = S$  and for  $x \neq y$ ,  $H(x, y)$  is a closed affine half space. For a nonempty closed convex  $C$ , we denote by  $Q_C(x, y, z)$  the projection, when it exists, of  $x$  onto  $H_C(x, y) \cap H_C(y, z)$  and  $Q$  the projection when  $C = \mathcal{H}$ , that is  $Q \stackrel{\text{def}}{=} Q_{\mathcal{H}}$ . As an intersection of two closed affine half spaces and a closed convex,  $H_C(x, y) \cap H_C(y, z)$  is a possibly empty closed convex.

It is easy to check, from the definition of  $H$ , that  $y$  is the projection of  $x$  onto  $H(x, y)$  and we therefore have  $Q(x, x, y) = P_{H(x, y)}x = y$ . Where  $P_C$  is the metric projection from  $\mathcal{H}$  onto  $C$ . Moreover, if  $y \in C$  then we also have that  $y$  is the projection of  $x$  onto  $H_C(x, y)$  which gives  $Q_C(x, x, y) = y$ .

The algorithm studied in [1] is the following

**Algorithm 1** *Given  $x_0 \in C$  and a sequence  $(T_n)_{n \geq 0}$  of mappings  $T_n : C \rightarrow \mathcal{H}$ , we consider the sequence  $(x_n)_{n \geq 0}$  generated by the following algorithm :*

$$x_{n+1} = Q_C(x_0, x_n, T_n x_n)$$

A very similar algorithm exists under the name of CQ algorithm [6, 8] :

**Algorithm 2** *Given  $x_0 \in C$ , we consider the sequence  $(x_n)_{n \geq 0}$  generated by the following algorithm :*

$$\begin{cases} y_n = R_n x_n, \\ C_n \stackrel{\text{def}}{=} \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\} , \\ D_n \stackrel{\text{def}}{=} \{z \in C \mid \langle x_n - z, x_0 - x_n \rangle \geq 0\} , \\ x_{n+1} = P_{(C_n \cap D_n)} x_0. \end{cases}$$

The link between the two algorithms is described by the following lemma.

**Lemma 1** *The sequence generated by Algorithm 2 coincides with the sequence given by  $x_{n+1} = Q_C(x_0, x_n, T_n x_n)$  with  $T_n \stackrel{\text{def}}{=} (R_n + Id)/2$ .*

*Proof :* Following [1], the proof easily follows from the equality

$$4 \langle z - Tx, x - Tx \rangle = \|Rx - z\|^2 - \|x - z\|^2.$$

□

The convergence of Algorithm 1 and therefore of Algorithm 2 when  $C = \mathcal{H}$  is studied in [1]. It relies on two requested properties of the sequence  $(T_n)_{n \geq 0}$ . First, the sequence  $(T_n)_{n \geq 0}$  must belong the  $\mathcal{T}$ -class which means that for all  $n \in \mathbb{N}$  we must have  $T_n \in \mathcal{T}$  where  $\mathcal{T}$  is defined as follows :

**Definition 2** *A mapping  $T : C \mapsto \mathcal{H}$  belongs to the  $\mathcal{T}_C$ -class if it is an element of the set  $\mathcal{T}_C$  :*

$$\mathcal{T}_C \stackrel{\text{def}}{=} \{T : C \mapsto \mathcal{H} \mid \text{dom}(T) = C \text{ and } (\forall x \in C) \text{Fix}(T) \subset H(x, Tx)\}.$$

When  $C = \mathcal{H}$ , we use the notation  $\mathcal{T} = \mathcal{T}_{\mathcal{H}}$ . Second, the sequence  $(T_n)_{n \geq 0}$  must be coherent as defined below.

**Definition 3** [1] *A sequence  $(T_n)_{n \geq 0}$  such that  $T_n \in \mathcal{T}_C$  is coherent if for every bounded sequence  $\{z_n\}_{n \geq 0} \in C$  the following holds :*

$$\begin{cases} \sum_{n \geq 0} \|z_{n+1} - z_n\|^2 < \infty \\ \sum_{n \geq 0} \|z_n - T_n z_n\|^2 < \infty \end{cases} \Rightarrow \mathcal{M}(z_n)_{n \geq 0} \subset \bigcap_{n \geq 0} \text{Fix}(T_n) \quad (3)$$

where  $\mathcal{M}(z_n)_{n \geq 0}$  is the set of weak cluster points of the sequence  $(z_n)_{n \geq 0}$ .

**Theorem 4** [1, Theorem 4.2] *Suppose that  $C = \mathcal{H}$  and the  $\mathcal{T}_C$ -class sequence  $(T_n)_{n \geq 0}$  is coherent. Then, for an arbitrary orbit of Algorithm 1, exactly one of the following alternatives holds :*

- (a)  $F \neq \emptyset$  and  $x_n \rightarrow_n P_F x_0$  ;
- (b)  $F = \emptyset$  and  $x_n \rightarrow_n +\infty$  ;
- (c)  $F = \emptyset$  and the algorithm terminates,

where the set  $F$  is defined by  $F \stackrel{\text{def}}{=} \bigcap_{n \geq 0} \text{Fix}(T_n)$ .

**Remark 5** *In the previous proof, it is supposed that  $C = \mathcal{H}$ . If  $C$  is a nonempty closed convex subset of  $\mathcal{H}$ , Theorem 4 (a) remains valid.*

In [9] another iterative algorithm called the *shrinking projection method* is studied. Using our notation it can be rephrased as follows :

**Algorithm 3** Given  $x_0 \in C$  and  $C_0 \stackrel{\text{def}}{=} C$ , we consider the sequence  $(x_n)_{n \geq 0}$  (when it exists) generated by the following algorithm :

$$\begin{cases} C_{n+1} \stackrel{\text{def}}{=} C_n \cap H(x_n, T_n x_n) & \text{with } T_n \stackrel{\text{def}}{=} (R_n + Id)/2, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases}$$

The previous algorithm is stopped once  $C_n = \emptyset$ . One of the results of this paper is the proof that the convergence of Algorithm 3 is governed by the same rules as for the convergence of Algorithm 1.

**Theorem 6** Suppose that the  $\mathcal{T}_C$ -class sequence  $(T_n)_{n \in \mathbb{N}}$  is coherent and let

$$F \stackrel{\text{def}}{=} \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n).$$

Then, if  $F \neq \emptyset$  the sequence  $(x_n)_{n \geq 0}$  produced by Algorithm 3 and Algorithm 1 converges to  $P_F x_0$ .

*Proof :* As pointed out in the introduction the case of Algorithm 1 when  $C = \mathcal{H}$  is proved in Theorem 4. The extension to the case of a closed nonempty subset  $C$  of  $\mathcal{H}$  is straightforward and we will not give an explicit proof. The proof of the case of Algorithm 3 is postponed to Section 4.  $\square$

**Remark 7** The first condition for the convergence is the fact that the sequence  $(T_n)_{n \geq 0}$  must belong to the  $\mathcal{T}_C$ -class. Note that by [1, Proposition 2.3]  $T \in \mathcal{T}$  iff the mapping  $2T - Id$  is quasi nonexpansive and  $\text{dom}(T) = \mathcal{H}$ . The equivalence remains true for  $\mathcal{T}_C$ -class if  $\text{dom}(T) = \mathcal{H}$  is replaced by  $\text{dom}(T) = C$ .

Thus, if  $T_n \stackrel{\text{def}}{=} (R_n + Id)/2$ , a necessary and sufficient condition for the sequence  $(T_n)_{n \geq 0}$  to belong to the  $\mathcal{T}_C$ -class is that the sequence  $(R_n)_{n \geq 0}$  is a sequence of quasi nonexpansive mappings.

**Remark 8** Moreover, it is a well known fact [3, Theorem 12.1] that  $2T - Id$  is nonexpansive iff  $T$  is firmly nonexpansive. Thus, a sufficient condition for the mapping  $T$  to belong to the  $\mathcal{T}_C$ -class is that  $T$  is a firmly nonexpansive mapping, i.e :

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle \quad \forall (x, y) \in C^2 \quad (4)$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(T - Id)x - (T - Id)y\|^2 \quad \forall (x, y) \in C^2. \quad (5)$$

We recall here the definition of the NST-condition (I) [5]. Let  $(T_n)_{n \geq 0}$  and  $\mathcal{F}$  be two families of nonexpansive mappings of  $C$  into itself such that

$$\emptyset \neq \text{Fix}(\mathcal{F}) \stackrel{\text{def}}{=} \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n),$$

where  $\text{Fix}(\mathcal{F})$  is the set of all common fixed points of mappings from the family  $\mathcal{F}$ .

**Definition 9** The sequence  $(T_n)_{n \geq 0}$  of mappings is said to satisfy the NST-condition (I) with  $\mathcal{F}$  if, for each bounded sequence  $(z_n)_{n \geq 0} \subset C$ , we have that  $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$  implies that  $\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0$  for all  $T \in \mathcal{F}$ .

**Remark 10** Suppose that  $\mathcal{F}$  is a family of nonexpansive mappings. It is easy to see that a sequence  $(T_n)_{n \geq 0}$  of mappings satisfying a NST-condition (I) with  $\mathcal{F}$  is coherent. Indeed, from a demi-closed principle or using [9, Lemma 3.1] if  $\|x_n - T x_n\| \rightarrow 0$  for all  $T \in \mathcal{T}$  then  $\mathcal{M}(x_n)_{n \geq 0} \subset \text{Fix}(\{T\}_{T \in \mathcal{T}})$ .

### 3 Coherent sequences of mappings

We consider here Algorithms 1 and 3 for a sequence of mappings  $(R_n)_{n \geq 0}$  built by  $N$  level iterations. Our aim is to give conditions under which the sequence  $(R_n)_{n \geq 0}$  or equivalently  $(T_n)_{n \geq 0} \stackrel{\text{def}}{=} (R_n + Id)/2$  is coherent<sup>1</sup> and apply Theorem 6 to get convergence results.

Let  $N \geq 1$  and  $(T_n^{(j)})_{n \geq 0} : C \rightarrow \mathcal{H}$  for  $1 \leq j \leq N$  be a finite set of sequences of nonexpansive mappings. Given also a family of sequences of real parameters  $(\alpha_n^{(j)})_{n \geq 0}$  for  $1 \leq j \leq N$ , we define new sequences  $(\Gamma_n^{(j)})_{n \geq 0} : C \rightarrow \mathcal{H}$  by the recursive equations :

$$\Gamma_n^{(j)} x \stackrel{\text{def}}{=} \alpha_n^{(j)} x + (1 - \alpha_n^{(j)}) T_n^{(j)} \Gamma_n^{(j+1)} x \quad \text{and} \quad \Gamma_n^{(N+1)} x \stackrel{\text{def}}{=} x \quad (6)$$

**H<sub>α</sub>** : We will assume that the sequences of real parameters  $(\alpha_n^{(j)})_{n \geq 0}$  satisfy the following condition : for  $2 \leq j \leq N$  and for all  $n \in \mathbb{N}$  we have  $\alpha_n^{(j)} \in (a, b)$  with  $0 < a < b < 1$  and  $\alpha_n^{(1)} \in [0, b)$ .

Using the sequence of mappings  $R_n \stackrel{\text{def}}{=} \Gamma_n^{(1)}$  in Algorithms 1 and 3 gives  $N$  level algorithms. We will consider the following specific examples :

**H<sub>1</sub>** Each sequence  $(T_n^{(j)})_{n \geq 0}$  is constant, i.e  $T_n^{(j)} = T^{(j)}$  for  $1 \leq j \leq N$  and  $F \stackrel{\text{def}}{=} \text{Fix}(\{T^{(j)}, 1 \leq j \leq N\})$  is nonempty.

**H<sub>2</sub>** The  $(T_n^{(j)})_{n \geq 0}$  sequences for  $1 \leq j \leq N$  are given by  $T_n^{(j)} = T^{(j)}(t_n)$ , where  $\{T^{(j)}(t) : t \geq 0\}$  is a finite set of given semigroups and  $(t_n)_{n \geq 0}$  is a sequence of real numbers such that  $\liminf_n t_n = 0$ ,  $\limsup_n t_n > 0$  and  $\lim_n (t_{n+1} - t_n) = 0$ . We assume that  $F \stackrel{\text{def}}{=} \text{Fix}(\{T^{(j)}(t), 1 \leq j \leq N, t \geq 0\})$  is nonempty.

**H<sub>3</sub>** The  $(T_n^{(j)})_{n \geq 0}$  sequences for  $1 \leq j \leq N$  are given by

$$T_n^{(j)} x = \frac{1}{t_n} \int_0^{t_n} T^{(j)}(s) x ds, \quad (7)$$

where  $\{T^{(j)}(t) : t \geq 0\}$  is a finite set of given semigroups and  $(t_n)_{n \geq 0}$  is a positive divergent sequence of real numbers. We assume that  $F \stackrel{\text{def}}{=} \text{Fix}(\{T^{(j)}(t), 1 \leq j \leq N, t \geq 0\})$  is nonempty.

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<sup>1</sup>By [1, Proposition 4.5] if  $(T_n)_{n \geq 0} \in \mathcal{T}$  and  $T'_n \stackrel{\text{def}}{=} Id + \lambda_n (T_n - Id)$  with  $\lambda_n \in [\delta, 1]$  and  $\delta \in ]0, 1]$ . Then  $(T_n)_{n \geq 0}$  is coherent iff  $(T'_n)_{n \geq 0}$  is coherent.

**Theorem 11** *Given a finite set of  $N$  nonexpansive sequences  $(T_n^{(j)})_{n \geq 0}$  satisfying  $\mathbf{H}_1$ ,  $\mathbf{H}_2$ , or  $\mathbf{H}_3$ . The sequence  $(x_n)_{n \geq 0}$  produced by Algorithm 1 and Algorithm 3 with  $R_n \stackrel{\text{def}}{=} \Gamma_n^{(1)}$  and  $(T_n)_{n \geq 0} \stackrel{\text{def}}{=} (R_n + Id)/2$  converges to  $P_F x_0$ . The mappings  $\Gamma_n^{(j)}$  being defined by equation (6) with parameters  $\alpha_n^{(j)}$  satisfying  $\mathbf{H}_\alpha$ .*

*Proof :* The proof is obtained by showing that the sequence of mappings  $(T_n)_{n \geq 0}$  is coherent in each given case and by applying Theorem 6 to conclude. The coherence is proved in the sequel in Proposition 15 for the case  $\mathbf{H}_1$ , in Proposition 17 for the case  $\mathbf{H}_2$  and in Proposition 19 for the case  $\mathbf{H}_3$ .  $\square$

We start here by a set of lemmata which are common to all cases.

**Lemma 12** *Let  $T$  be a  $F$ -quasi nonexpansive mapping and for  $\beta \in (0, 1)$  the mapping  $T_\beta \stackrel{\text{def}}{=} \beta Id + (1 - \beta)T$ . For  $p \in F$  and all  $x \in H$  we have :*

$$\beta(1 - \beta)\|x - Tx\|^2 \leq 2(\|x - p\| - \|T_\beta x - p\|)\|x - p\| \quad (8)$$

*Proof :* For  $p \in F$  and all  $x \in H$  we have :

$$\begin{aligned} \|T_\beta x - p\|^2 &= \|\beta(x - p) + (1 - \beta)(Tx - p)\|^2 \\ &= \beta\|x - p\|^2 + (1 - \beta)\|Tx - p\|^2 - \beta(1 - \beta)\|Tx - x\|^2 \\ &\leq \|x - p\|^2 - \beta(1 - \beta)\|Tx - x\|^2. \end{aligned}$$

We thus obtain

$$\begin{aligned} \beta(1 - \beta)\|Tx - x\|^2 &\leq (\|x - p\| - \|T_\beta x - p\|)(\|x - p\| + \|T_\beta x - p\|) \\ &\leq 2(\|x - p\| - \|T_\beta x - p\|)\|x - p\|. \end{aligned}$$

$\square$

**Lemma 13** *Let  $T$  a  $F$ -quasi nonexpansive mapping. For  $\beta \in (0, 1)$  we define the mapping  $T_\beta \stackrel{\text{def}}{=} \beta Id + (1 - \beta)T$ . For  $p \in F$ , all  $x \in H$  and  $S$  a  $F$ -quasi nonexpansive mapping, we have :*

$$\beta(1 - \beta)\|x - Tx\|^2 \leq 2\|x - ST_\beta x\|\|x - p\|. \quad (9)$$

*If moreover  $S$  is nonexpansive we also have :*

$$\|x - Sx\| \leq \|x - ST_\beta x\| + \|Tx - x\|. \quad (10)$$

*Proof :* For  $p \in F$  and all  $x \in H$  we have :

$$\begin{aligned}\|x - p\| &\leq \|x - ST_\beta x\| + \|ST_\beta x - p\| \\ &\leq \|x - ST_\beta x\| + \|T_\beta x - p\|.\end{aligned}$$

We thus have  $\|x - p\| - \|T_\beta x - p\| \leq \|x - ST_\beta x\|$  which combined with Lemma 12 gives equation (9).

Now if  $S$  is nonexpansive,

$$\begin{aligned}\|x - Sx\| &\leq \|x - ST_\beta x\| + \|ST_\beta x - Sx\| \leq \|x - ST_\beta x\| + \|T_\beta x - x\| \\ &\leq \|x - ST_\beta x\| + (1 - \beta)\|Tx - x\| \leq \|x - ST_\beta x\| + \|Tx - x\|.\end{aligned}$$

□

**Lemma 14** Suppose that  $F \stackrel{\text{def}}{=} \bigcap_{\{n \in \mathbb{N}; 1 \leq j \leq N\}} \text{Fix}(T_n^{(j)})$  is not empty suppose that for a bounded sequence  $(x_n)_{n \geq 0}$  and a fixed value of  $j$  we have  $\|x_n - T_n^{(j)} \Gamma_n^{(j+1)} x_n\| \rightarrow 0$ . Moreover, suppose that for  $2 \leq j \leq N$  and all  $n \in \mathbb{N}$  we have  $\alpha_n^{(j)} \in (a, b)$  with  $0 < a < b < 1$ . Then for all  $k \geq j$  we have  $\|x_n - T_n^{(k)} x_n\| \rightarrow 0$ .

*Proof :* Note first that the sequences  $(T^{(j)})_{1 \leq j \leq N}$  and  $(\Gamma^{(j)})_{1 \leq j \leq N+1}$  are composed of nonexpansive mappings. Indeed the composition of nonexpansive mappings is nonexpansive and for  $\beta \in (0, 1)$   $\beta Id + (1 - \beta)S$  is nonexpansive when  $S$  is nonexpansive. The sequences are also  $F$ -quasi nonexpansive since it is straightforward that  $F \subset \text{Fix}(\Gamma_n^{(j)})$  for all  $j \in [1, N]$  and  $n \in \mathbb{N}$  and if  $S$  is nonexpansive it is also  $\text{Fix}(S)$ -quasi nonexpansive.

The proof then follows by backward induction on  $j$ . Assume that the result is true for  $j + 1$  then we will prove that it is true for  $j$ . Using the definition of  $\Gamma_n^{(j+1)}$  and using equation (9) for  $p \in F$ ,  $S = T_n^{(j)}$ ,  $T = T_n^{(j+1)} \Gamma_n^{(j+2)}$  and  $\beta = \alpha_n^{(j+1)}$  (we thus have  $T_\beta = \Gamma_n^{(j+1)}$ ) we obtain :

$$\alpha_n^{(j+1)}(1 - \alpha_n^{(j+1)})\|x_n - T_n^{(j+1)} \Gamma_n^{(j+2)} x_n\|^2 \leq 2\|x_n - T_n^{(j)} \Gamma_n^{(j+1)} x_n\|\|x_n - p\| \quad (11)$$

We thus obtain that  $\|x_n - T_n^{(j+1)} \Gamma_n^{(j+2)} x_n\| \rightarrow 0$  and by induction hypothesis we obtain  $\|x_n - T_n^{(k)} x_n\| \rightarrow 0$  for  $k \geq j + 1$ . Now using equation (10) with  $S \stackrel{\text{def}}{=} T_n^{(j)}$ ,  $T \stackrel{\text{def}}{=} T_n^{(j+1)} \Gamma_n^{(j+2)}$  and  $\beta = \alpha_n^{(j+1)}$  we get :

$$\|x_n - T_n^{(j)} x_n\| \leq \|x_n - T_n^{(j)} \Gamma_n^{(j+1)} x_n\| + \|T_n^{(j+1)} \Gamma_n^{(j+2)} x_n - x_n\| \quad (12)$$

and the result follows for  $j$ . □

### 3.1 The case $\mathbf{H}_1$

**Proposition 15** In the case  $\mathbf{H}_1$ , the sequence  $(R_n)_{n \geq 0}$ , defined by  $R_n \stackrel{\text{def}}{=} \Gamma_n^{(1)}$  with parameters satisfying  $\mathbf{H}_\alpha$ , satisfy the NST-condition(I) with  $\mathcal{F} \stackrel{\text{def}}{=} \text{Fix}\{T^{(j)}\}_{1 \leq j \leq N}$  and the sequence  $T_n = (R_n + Id)/2$  is a  $\mathcal{T}_C$ -class and coherent sequence.



*Proof :* We have  $\|x_n - R_n x_n\| = \|x_n - T_n^{(1)} \Gamma_n^{(2)} x_n\| (1 - \alpha_n^{(1)})$ . Thus, if for each bounded sequence  $(x_n)_{n \geq 0}$   $\|x_n - R_n x_n\| \mapsto 0$  we also have  $\|x_n - T_n^{(1)} \Gamma_n^{(2)} x_n\| \mapsto 0$  since  $(1 - \alpha_n^{(1)})$  is bounded from zero. Using Lemma 14 we have  $\|x_n - T^{(j)} x_n\| \mapsto 0$  for  $1 \leq j \leq N$  which gives use the NST-condition(I) with  $\mathcal{F}$ . Now we consider the sequence  $(T_n)_{n \geq 0}$ . The sequence belongs to the  $\mathcal{T}_C$ -class since  $2T_n - Id = R_n$  is nonexpansive and thus quasi nonexpansive. Now if  $\|x_n - T_n x_n\| \mapsto 0$  we also have  $\|x_n - R_n x_n\| \mapsto 0$  and thus using the NST-condition(I) we have  $\|x_n - T^{(j)} x_n\| \mapsto 0$  for  $1 \leq j \leq N$ . Since the  $T^{(j)}$  are nonexpansive they are also demi-closed [2, Lemma 4] and thus we must have  $\mathcal{M}(x_n)_{n \geq 0} \subset \text{Fix}(\{T^{(j)}, 1 \leq j \leq N\}) = \text{Fix}(\{T_n\}_{n \in \mathbb{N}})$ . The sequence  $(T_n)_{n \geq 0}$  is thus in the  $\mathcal{T}_C$ -class and coherent.  $\square$

**Remark 16** For  $N = 1$  we recover [9, Theorem 1.1] and [9, Theorem 4.1].

### 3.2 The case $\mathbf{H}_2$

Let  $\{T(t) : t \geq 0\}$  be a family of mappings from a subset  $C$  of  $\mathcal{H}$  into itself. We call it a nonexpansive semigroup on  $C$  if the following conditions are satisfied :

- (i)  $T(0)x = x$  for all  $x \in C$  ;
- (ii)  $T(s+t) = T(s)T(t)$  for all  $s, t \geq 0$  ;
- (iii) for each  $x \in C$  the mapping  $t \mapsto T(t)x$  is continuous ;
- (iv)  $\|T(t)x - T(t)y\| \leq \|x - y\|$  for all  $x, y \in C$  and  $t \geq 0$ .

**Proposition 17** In the case  $\mathbf{H}_2$ , the sequence  $(R_n)_{n \geq 0}$ , defined by  $R_n \stackrel{\text{def}}{=} \Gamma_n^{(1)}$  with parameters satisfying  $\mathbf{H}_\alpha$ , satisfy the NST-condition(I) with  $\mathcal{F} \stackrel{\text{def}}{=} \text{Fix}\{T^{(j)}(t)_{1 \leq j \leq N, t \geq 0}\}$  and the sequence  $T_n = (R_n + Id)/2$  is a  $\mathcal{T}_C$ -class and coherent sequence.

*Proof :* As in the proof of Proposition 15 we obtain that for each bounded sequence  $(x_n)_{n \geq 0}$  such that  $\|x_n - R_n x_n\| \mapsto 0$  we also have  $\|x_n - T^{(j)}(t_n)x_n\| \mapsto 0$  for  $1 \leq j \leq N$ . Now it is easy to prove that the weak cluster points of the sequence  $(x_n)_{n \geq 0}$  are in  $F$ . The proof for each fixed  $j$  is the same as in [7, Theorem 2.2, page 6]. We thus obtain the coherence of the sequence  $(T_n)_{n \geq 0}$ .  $\square$

**Remark 18** For  $N = 1$  we recover [7, Theorem 2.1] for Algorithm 3 and [7, Theorem 2.2] for Algorithm 1.

### 3.3 The case $\mathbf{H}_3$

**Proposition 19** In the case  $\mathbf{H}_3$ , the sequence  $(R_n)_{n \geq 0}$ , defined by  $R_n \stackrel{\text{def}}{=} \Gamma_n^{(1)}$  with parameters satisfying  $\mathbf{H}_\alpha$ , satisfy the NST-condition(I) with  $\mathcal{F} \stackrel{\text{def}}{=} \text{Fix}\{T^{(j)}(t)_{1 \leq j \leq N, t \geq 0}\}$  and the sequence  $T_n = (R_n + Id)/2$  is a  $\mathcal{T}_C$ -class and coherent sequence.

*Proof :* As in the proof of Proposition 15 we obtain that for each bounded sequence  $(x_n)_{n \geq 0}$  such that  $\|x_n - R_n x_n\| \mapsto 0$  we also have  $\|x_n - T^{(j)}(t_n)x_n\| \mapsto 0$  for  $1 \leq j \leq N$ . Now it is easy to prove that the weak cluster points of the sequence  $(x_n)_{n \geq 0}$  are in  $F$ . The proof for each fixed  $j$  is the same as in [6, Theorem 4.1]. For each fixed  $j$ , it is a consequence of the inequality [6, Equation (8)] :

$$\|T^{(j)}(s)x_n - x_n\| \leq 2\|T_n^{(j)}x_n - x_n\| + \|T(s)(T_n^{(j)}x_n) - T_n^{(j)}x_n\| \quad (13)$$

for every  $0 \leq s < +\infty$  and  $n \in \mathbb{N}$  with  $T_n^{(j)}$  and the fact that the right hand side of the above inequality goes to zero as  $n$  goes to infinity for a bounded sequence  $(x_n)_{n \geq 0}$  using [6, Lemma 2.1]. We thus obtain the coherence of the sequence  $(T_n)_{n \geq 0}$ .  $\square$

**Remark 20** For  $N = 1$  we recover [6, Theorem 4.1] for Algorithm 1 and [9, Theorem 4.4] for Algorithm 3.

## 4 Proof of Theorem 6

We prove here the strong convergence of Algorithm 3 for a  $\mathcal{T}_C$ -class sequence of coherent mappings. The proof follows the same steps as the proof of the convergence of Algorithm 1 in [1], we therefore give references to the original propositions.

The proof results from the next proposition and theorem in the following way. Let  $(x_n)_{n \geq 0}$  be an arbitrary orbit of Algorithm 3 and let  $F \stackrel{\text{def}}{=} \text{Fix}(\{T_n\}_{n \in \mathbb{N}})$ . If  $F \neq \emptyset$ , then by Proposition 21 (iv) the sequence is defined. By Theorem 22 (ii) the sequence is bounded. Thus (v) is fulfilled and by the coherence property we have  $\mathcal{M}(x_n)_{n \geq 0} \subset F$ . Then, by Theorem 22 (iv), the sequence strongly converges to  $P_F(x_0)$ .

**Proposition 21** [1, Proposition 3.4] Let  $(x_n)_{n \geq 0}$  be an arbitrary orbit of Algorithm 3. Then :

- (i) If  $x_{n+1}$  is defined then  $\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|$ .
- (ii) If  $x_n$  is defined then  $x_0 = x_n \iff x_n = x_{n-1} = \dots = x_0 \iff x_0 \in \bigcup_{k=0}^{n-1} \text{Fix}(T_k)$ .
- (iii) If  $(x_n)_{n \geq 0}$  is defined then  $(\|x_0 - x_n\|)_{n \in \mathbb{N}}$  is increasing.
- (iv)  $(x_n)_{n \geq 0}$  is defined if  $F \stackrel{\text{def}}{=} \text{Fix}(\{T_n\}_{n \in \mathbb{N}}) \neq \emptyset$ .

*Proof :* (i) : If  $x_{n+1}$  is defined we have  $x_{n+1} = P_{C_{n+1}}x_0$  and thus  $x_{n+1} \in C_{n+1} \subset C_n$  and since  $x_n = P_{C_n}x_0$  we have  $\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|$ . (ii) : The first equivalence follows from (i). The second one is proved by induction. Note first that  $H$  is such that  $y = P_{H(x,y)}x$ . Now for  $y \in C$ , we obtain also that  $y = P_{C \cap H(x,y)}x$ . for  $n = 1$ , we have  $x_1 = P_{C \cap H(x_0, T_0 x_0)}x_0 = T_0 x_0$  and thus  $x_1 = x_0 \iff x_0 \in \text{Fix}(T_0)$ . Now assume that the equivalence is fulfilled for  $n$ .

We have

$$x_{n+1} = x_n = \dots = x_0 \iff \begin{cases} x_0 & \in \cup_{k=0}^{n-1} \text{Fix}(T_k) \\ x_0 & = x_{n+1} = P_{C \cap \bigcap_{k=0}^n H(x_k, T_k x_k)} \\ & = P_{C \cap H(x_0, T_n x_0)} = T_n x_0. \end{cases}$$

(iii) follows from (i). (iv) : The algorithm is defined if  $C_n \neq \emptyset$  for all  $n \in \mathbb{N}$ . Thus it is enough to prove that  $C \cap (\bigcap_{n \in \mathbb{N}} H(x_n, T_n x_n)) \neq \emptyset$ . By definition of the  $\mathcal{T}_C$  class we have  $\text{Fix}(T_n) \subset C \cap H(x_n, T_n x_n)$  and the result follows.  $\square$

**Theorem 22** ([1, Theorem 3.5]) *Let  $(x_n)_{n \geq 0}$  be an arbitrary orbit of Algorithm 3 and let  $F \stackrel{\text{def}}{=} \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n)$ . Then*

- (i) *If  $(x_n)_{n \geq 0}$  is defined then :  $(x_n)_{n \geq 0}$  is bounded  $\iff (\|x_0 - x_n\|)_{n \in \mathbb{N}}$  converges.*
- (ii) *If  $F \neq \emptyset$ , then  $(x_n)_{n \geq 0}$  is bounded and  $(\forall n \in \mathbb{N}) x_n \in F \iff x_n = P_F(x_0)$ .*
- (iii) *If  $F \neq \emptyset$ , then  $(\|x_0 - x_n\|)_{n \in \mathbb{N}}$  converges and  $\lim_n \|x_0 - x_n\| \leq \|x_0 - P_F x_0\|$ .*
- (iv) *If  $F \neq \emptyset$ , then :  $\lim_n x_n = P_F(x_0) \iff \mathcal{M}(x_n)_{n \in \mathbb{N}} \subset F$ .*
- (v) *If  $(x_n)_{n \geq 0}$  is defined and bounded then  $\sum_{n \geq 0} \|x_{n+1} - x_n\|^2 < +\infty$  and  $\sum_{n \geq 0} \|x_n - T_n x_n\|^2 < +\infty$ .*

*Proof :* (i) follows from Proposition 21 (i). (ii) : If  $F \neq \emptyset$  then by Proposition 21 (iv) the sequence is defined. We have  $F \subset C \cap (\bigcap_{n \in \mathbb{N}} H(x_n, T_n x_n))$  and thus  $F \subset C_n$ . Now, from  $P_F(x_0) \in C_n$  and  $x_n = P_{C_n} x_0$  we obtain  $\|x_n - x_0\| \leq \|x_0 - P_F(x_0)\|$  and (ii) follows. (iii) follows from (i), (ii) and the previous inequality. (iv) : The forward implication is trivial. For the reverse implication, the proof exactly follows (iv) of [1, Theorem 3.5] since it does not involve  $C$ . (v) : From  $x_n = P_{C_n} x_0$  and  $x_{n+1} \in C_n$  we obtain :

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0.$$

We thus have :

$$\begin{aligned} \|x_{n+1} - x_n\|^2 & \leq \|x_{n+1} - x_n\|^2 + 2 \langle x_{n+1} - x_n, x_n - x_0 \rangle \\ & \leq \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2. \end{aligned} \quad (14)$$

Hence  $\sum_{n \geq 0} \|x_{n+1} - x_n\|^2 \leq \sup_{n \in \mathbb{N}} \|x_0 - x_n\|^2 < +\infty$  since  $(x_n)_{n \geq 0}$  is bounded. For all  $n \in \mathbb{N}$  we have  $x_{n+1} \in H(x_n, T_n x_n)$ , which implies,

$$\begin{aligned} \|x_{n+1} - x_n\|^2 & = \|x_{n+1} - T_n x_n\|^2 - 2 \langle x_{n+1} - T_n x_n, x_n - T_n x_n \rangle \\ & + \|x_n - T_n x_n\|^2 \\ & \geq \|x_n - T_n x_n\|^2, \end{aligned} \quad (15)$$

and we therefore obtain  $\sum_{n \geq 0} \|x_n - T_n x_n\|^2 < +\infty$ .  $\square$

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